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Generic extensions of finite-valued u.s.c. selections [☆]

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Abstract

As a rule, most of the classical Michael-type selection theorems are analogues and, in some respects, generalizations of ordinary extension theorems. In this paper we show that the existence of set-valued u.s.c. selections for l.s.c. mappings is not related to the "usual" mapping-extension problem for u.s.c. mappings. In view of that, the paper is especially devoted to a proper notion of extending u.s.c. mappings that agrees well with the existing selection results. On this base new selection theorems dealing with controlled u.s.c. "extensions" of partial u.s.c. selections are obtained. Possible applications are illustrated in the dimension theory of normal spaces. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let X and Y be topological spaces, and let 2^Y stand for the family of non-empty subsets of Y. We write

$$\mathcal{F}(Y)=\left\{S\in 2^Y\colon\ S \text{ is closed}\right\},\qquad \mathcal{C}(Y)=\left\{S\in \mathcal{F}(Y)\colon\ S \text{ is compact}\right\}$$

and, for every natural number $m\in\mathbb{N}$,

$$\mathcal{C}_m(Y)=\left\{S\in\mathcal{F}(Y)\colon\ |S|\leqslant m\right\}.$$

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A set-valued mapping $\Phi : X \to 2^Y$ is *lower semi-continuous* (*upper semi-continuous*), or l.s.c. (u.s.c.), if the set

$$\Phi^{-1}(U) = \{x \in X\colon\ \Phi(x) \cap U \neq \emptyset\}$$

is open (respectively, closed) in X for every open (respectively, closed) subset U of Y. A set-valued mapping $\varphi : X \to 2^Y$ is a *selection* for $\Phi : X \to 2^Y$ if $\varphi(x) \subset \Phi(x)$ for every $x \in X$.

A starting point of the paper is given by two well-known results concerning the existence of u.s.c. selections for l.s.c. mappings. The first of them is due to Michael and a part of it sounds as follows.

Theorem 1.1 (Michael [14]). *Let X be a paracompact space, Y be a completely metrizable space, and let $\Phi : X \to \mathcal{F}(Y)$ be an l.s.c. mapping. Then Φ admits a u.s.c. selection $\varphi : X \to \mathcal{C}(Y)$.*

The second one is the following dimension-type version of Theorem 1.1.

Theorem 1.2 (Choban [2]). *Let X be a paracompact space such that* $\dim(X) \leqslant n$, *Y be a completely metrizable space, and let $\Phi : X \to \mathcal{F}(Y)$ be an l.s.c. mapping. Then Φ admits a u.s.c. selection $\varphi : X \to \mathcal{C}_{n+1}(Y)$.*

It should be mentioned here that, as a rule, this type of selection theorems are analogues and, in most respects, generalizations of ordinary extension theorems. However, in contrast to this, Theorems 1.1 and 1.2 do not generalize any extension result which leads us to the following natural question.

Question 1.3. Under the assumptions of Theorem 1.1 (respectively, Theorem 1.2) let, in addition, A be a closed subset of X and let $\theta : A \to \mathcal{C}(Y)$ (respectively, $\theta : A \to \mathcal{C}_{n+1}(Y)$) be a u.s.c. selection for $\Phi|A$. Does there exist a u.s.c. selection $\varphi : X \to \mathcal{C}(Y)$ (respectively, $\varphi : X \to \mathcal{C}_{n+1}(Y)$) for Φ which extends θ, i.e., $\varphi|A = \theta$?

As it is shown below, the answer to this question is "no" which indicates that a principal difference between "the finding a u.s.c. selection" and "the extending a u.s.c. selection" does exist.

Proposition 1.4. *Let $Y = \{0, 1\}$ be the discrete two-point space, and let X be a T_1-space such that, whenever $A \subset X$ is closed, every u.s.c. mapping $\theta : A \to 2^Y$ can be extended to a u.s.c. mapping $\varphi : X \to 2^Y$. Then X is hereditarily normal.*

Proof. Let $A_0, A_1 \subset X$ be such that $\mathrm{cl}(A_i) \cap A_{1-i} = \emptyset$, $i = 0, 1$. We look for open sets $U_i \subset X$ such that $U_i \cap U_{1-i} = \emptyset$ and $A_i \subset U_i$, $i = 0, 1$. To this end, set $A = \mathrm{cl}(A_0) \cup \mathrm{cl}(A_1)$ and then define a u.s.c. mapping $\theta : A \to 2^Y$ by $\theta(x) = \{i \in Y\colon\ x \in \mathrm{cl}(A_i)\}$, $x \in A$. Then $U_i = X \setminus \varphi^{-1}(\{1 - i\})$, $i = 0, 1$, are as required, where $\varphi : X \to 2^Y$ is a u.s.c. extension of θ. $\square$

Proposition 1.4 is also important from another point of view. It suggests the interesting problem given by whether the answer to Question 1.3 becomes "yes" if the restriction on X is strengthened to "paracompact and hereditarily normal". Recently a positive answer was given by Shishkov [18].

In the present paper, we regard another approach to "extensions" of set-valued mappings. Concerning especially u.s.c. ones, we shall slightly change our point of view allowing a *controlled expansion* of their values which reflects to our primary question of interest in the following way.

Question 1.3′. Under the assumptions of Theorem 1.1 (respectively, Theorem 1.2) let, in addition, A be a closed subset of X and let $\theta : A \to \mathcal{C}(Y)$ (respectively, $\theta : A \to \mathcal{C}_{n+1}(Y)$) be a u.s.c. selection for $\Phi|A$. Does there exist a u.s.c. selection $\varphi : X \to \mathcal{C}(Y)$ (respectively, $\varphi : X \to \mathcal{C}_{n+1}(Y)$) for Φ such that $\theta(x) \subset \varphi(x)$ for all $x \in A$?

First of all, let us observe that such an understanding of u.s.c. "extensions" of u.s.c. selections is in a good accordance with Theorem 1.1.

Proposition 1.5. *For a set-valued mapping $\Phi : X \to \mathcal{F}(Y)$ the following conditions are equivalent*:

(a) *Φ admits a u.s.c. selection $\psi : X \to \mathcal{C}(Y)$.*

(b) *If $A \subset X$ is closed and $\theta : A \to \mathcal{C}(Y)$ is a u.s.c. selection for $\Phi|A$, then there exists a u.s.c. selection $\varphi : X \to \mathcal{C}(Y)$ for Φ such that $\theta(x) \subset \varphi(x)$ for all $x \in A$.*

Proof. (b) $\Rightarrow$ (a) is obvious. As for (a) $\Rightarrow$ (b), let $\psi : X \to \mathcal{C}(Y)$ be as in (a) and let $\theta : A \to \mathcal{C}(Y)$ be a u.s.c. selection for $\Phi|A$ for some closed $A \subset X$. Then the set-valued mapping $\varphi : X \to \mathcal{C}(Y)$, defined by $\varphi(x) = \psi(x) \cup \theta(x)$ if $x \in A$ and $\varphi(x) = \psi(x)$ otherwise, is as required in (b). □

The main purpose of this paper is to present a complete solution of Question 1.3′. In fact, Proposition 1.5 is the first and most trivial part in this answering. In view of that we shall henceforth restrict our attention only to finite-valued u.s.c. selections. Among the theorems we shall prove in this direction, the following two probably have the greatest general interest.

Theorem 1.6. *Let X be a paracompact space, $A \subset X$ closed, Y a completely metrizable space, $\Phi : X \to \mathcal{F}(Y)$ l.s.c., and let, for some $m \in \mathbb{N}$, $\theta : A \to \mathcal{C}_m(Y)$ be a u.s.c. selection for $\Phi|A$. Then Φ admits a u.s.c. selection $\varphi : X \to \mathcal{C}(Y)$ such that $\theta(x) \subset \varphi(x)$ and $|\varphi(x)| \leqslant m$ for all $x \in A$.*

Theorem 1.7. *Let X be a paracompact space, $A \subset X$ closed with $\dim_X(X\setminus A) \leqslant n$, Y a completely metrizable space, $\Phi : X \to \mathcal{F}(Y)$ l.s.c., and let, for some $m \leqslant n+1$, $\theta : A \to \mathcal{C}_m(Y)$ be a u.s.c. selection for $\Phi|A$. Then Φ admits a u.s.c. selection $\varphi : X \to \mathcal{C}_{n+1}(Y)$ such that $\theta(x) \subset \varphi(x)$ and $|\varphi(x)| \leqslant m$ for all $x \in A$.*

Here, $\dim_X(X\setminus A) \leqslant n$ means that $\dim(S) \leqslant n$ for every $S \subset X\setminus A$ which is closed in X.

The technique developed for proving Theorems 1.6 and 1.7 allows one to establish also natural their collectionwise normal versions (see Section 5). Some possible applications of these "selection–extension" results are presented in the last Section 6 of the paper. The following partial case of Theorem 6.1 illustrates this kind of applications.

Theorem 1.8. *For a normal space X and $n \geqslant 0$ the following conditions are equivalent*

(a) $\dim(X) \leqslant n$.

(b) *Whenever $A \subset X$ is closed and Y is a compact metric space, every u.s.c. $\theta : A \to \mathcal{C}_{n+1}(Y)$ is a selection for $\varphi|A$ for some u.s.c. $\varphi : X \to \mathcal{C}_{n+1}(Y)$.*

The paper is arranged as follows. Section 2 is devoted to special approximate representations of compact-valued u.s.c. selections. On this base, it is introduced in Section 3 a concept of a u.s.c. mapping being an *expansion* of another u.s.c. mapping and is established the existence of such expansions in a number of situations. A dimension-type improvement of these results is next obtained in Section 4. Theorems 1.6 and 1.7 are finally proved in Section 5. Section 6 is devoted to applications.

2. Decompositions of usco selections

Throughout this section, X is a topological space and (Y, d) is a metric space. A set-valued mapping $\theta : X \to 2^Y \cup \{\emptyset\}$ is *usco* if it is u.s.c. and each $\theta(x)$ is a non-empty compact subset of Y. For a subset $M \subset X$ and a collection $\mathcal{W}$ of subsets of X we use $\operatorname{Ord}(\mathcal{W}; M)$ to denote the *order* of $\mathcal{W}$ in M, i.e., the smallest cardinal number τ such that

$$\bigl|\{W \in \mathcal{W}\colon\ x \in W\}\bigr| \leqslant \tau \quad \text{for every } x \in M.$$

For $S \in 2^Y$ and $\varepsilon > 0$, we use $B^d_\varepsilon(S)$ to denote the ε-neighbourhood of S in (Y, d), i.e., $B^d_\varepsilon(S) = \{y \in Y\colon\ d(y, S) < \varepsilon\}$. Finally, let $\mathcal{T}(Y)$ denote the topology of Y and let $\mathcal{P}(X) = \mathcal{F}(X) \cup \{\emptyset\}$.

Definition 2.1. Let $\Phi : X \to \mathcal{F}(Y)$ be a set-valued mapping, $\mathcal{A}$ be a set, $p : \mathcal{A} \to \mathcal{P}(X)$, and let $t : \mathcal{A} \to \mathcal{T}(Y)$. We shall say that the triple $(p, t; \mathcal{A})$ is a $t(\mathcal{A})$-*approximate* selection for Φ (see [7,8]) if

(a) the indexed family $\{p(\alpha)\colon\ \alpha \in \mathcal{A}\}$ is a locally finite cover of X,

(b) the indexed family $\{t(\alpha)\colon\ \alpha \in \mathcal{A}\}$ is a locally finite cover of Y,

(c) $p(\alpha) \subset \Phi^{-1}(\operatorname{cl}(t(\alpha)))$ for every $\alpha \in \mathcal{A}$.

In the set

$$\Theta(\Phi) = \bigl\{(p, t; \mathcal{A})\colon\ (p, t; \mathcal{A}) \text{ is a } t(\mathcal{A})\text{-approximate selection for } \Phi\bigr\}$$

we define a partial order. Namely, for $(p, t; \mathcal{A}), (q, \ell; \mathcal{B}) \in \Theta(\Phi)$ and a map $r : \mathcal{A} \to \mathcal{B}$, we shall write that $(p, t; \mathcal{A}) \ll_r (q, \ell; \mathcal{B})$ if, for every $\beta \in \mathcal{B}$,

$$q(\beta) \supset \bigcup \bigl\{p(\alpha)\colon\ \alpha \in r^{-1}(\beta)\bigr\} \quad \text{and} \quad \ell(\beta) = \bigcup \bigl\{t(\alpha)\colon\ \alpha \in r^{-1}(\beta)\bigr\}.$$

Concerning the set $\Theta(\Phi)$ we also define *Order* of $(p,t;\mathcal{A}) \in \Theta(\Phi)$ on a subset $M \subset X$ by letting

$$\mathrm{Ord}_M(p,t;\mathcal{A}) = \mathrm{Ord}\bigl(\bigl\{p(\alpha)\colon\ \alpha \in \mathcal{A}\bigr\};\, M\bigr),$$

and the *mesh* of $(p,t;\mathcal{A})$ by

$$\mathrm{mesh}(p,t;\mathcal{A}) = \sup\bigl\{\mathrm{diam}(t(\alpha))\colon\ \alpha \in \mathcal{A}\bigr\}.$$

To every $(p,t;\mathcal{A}) \in \Theta(\Phi)$ we associate a u.s.c. mapping $[p,t;\mathcal{A}] : X \to \mathcal{F}(Y)$ defined by

$$[p,t;\mathcal{A}](x) = \bigcup \bigl\{\mathrm{cl}(t(\alpha))\colon\ \alpha \in \mathcal{A} \text{ and } x \in p(\alpha)\bigr\}.$$

Finally, we need also the following special subset of $\Theta(\Phi)$:

$$\Omega(\Phi) = \bigl\{(p,t;\mathcal{A}) \in \Theta(\Phi)\colon\ p(\alpha) \subset \Phi^{-1}(t(\alpha)) \text{ for every } \alpha \in \mathcal{A}\bigr\}.$$

Definition 2.2. Let $\theta : X \to 2^Y \cup \{\emptyset\}$, and let $\Gamma \subset \Theta(\Phi)$. We shall say that $\langle (p_k,t_k;\mathcal{A}_k), r_k\rangle_{k\in\mathbb{N}}$ is a *Γ-decomposition* of θ provided $\{(p_k,t_k;\mathcal{A}_k)\colon\ k \in \mathbb{N}\} \subset \Gamma$, $r_k : \mathcal{A}_{k+1} \to \mathcal{A}_k$, $k \in \mathbb{N}$, and

(a) $(p_{k+1},t_{k+1};\mathcal{A}_{k+1}) \ll_{r_k} (p_k,t_k;\mathcal{A}_k)$, $k \in \mathbb{N}$,

(b) $\lim_{k\to\infty} \mathrm{mesh}(p_k,t_k;\mathcal{A}_k) = 0$,

(c) $\theta(x) = \bigcap\{[p_k,t_k;\mathcal{A}_k](x)\colon\ k \in \mathbb{N}\}$, $x \in X$.

The main result of this section reads now as follows.

Theorem 2.3. *Let (Y,d) be a complete metric space, and let $\Phi : X \to \mathcal{F}(Y)$. For a mapping $\theta : X \to 2^Y \cup \{\emptyset\}$ the following conditions are equivalent*:

(a) *θ is an usco selection for Φ.*

(b) *θ admits an $\Omega(\Phi)$-decomposition.*

In preparation for the proof of Theorem 2.3 we first need the following lemma.

Lemma 2.4. *Let (Y,d) be a metric space, $\Phi : X \to \mathcal{F}(Y)$, and let θ be an usco selection for Φ. Then θ admits an $\Omega(\Phi)$-decomposition $\langle (p_k,t_k;\mathcal{A}_k), r_k\rangle_{k\in\mathbb{N}}$ and a $\Theta(\Phi)$-decomposition $\langle (p_k,s_k;\mathcal{A}_k), r_k\rangle_{k\in\mathbb{N}}$ such that, for all k and $\alpha \in \mathcal{A}_k$,*

$$\mathrm{cl}\bigl(s_k(\alpha)\bigr) \subset t_k(\alpha) \quad \textit{and} \quad p_k(\alpha) = \theta^{-1}\bigl(\mathrm{cl}\bigl(s_k(\alpha)\bigr)\bigr).$$

Proof. Since (Y,d) is a metric space, there is a locally finite sieve $(\{U_\alpha\colon\ \alpha \in \mathcal{A}_k\}, r_k)$ on Y such that $\mathrm{diam}(U_\alpha) < 1/k$ for all $\alpha \in \mathcal{A}_k$. This means that $\{U_\alpha\colon\ \alpha \in \mathcal{A}_k\}_{k\in\mathbb{N}}$ is a sequence of locally finite open covers of Y (with disjoint $\mathcal{A}_k$'s), and $r_k : \mathcal{A}_{k+1} \to \mathcal{A}_k$ are maps such that, for all k and $\alpha \in \mathcal{A}_k$,

$$U_\alpha = \bigcup \bigl\{U_\beta\colon\ \beta \in r_k^{-1}(\alpha)\bigr\},$$

[1,15]. By [15, Lemma 2.2], there also exists a sieve $(\{V_\alpha\colon\ \alpha \in \mathcal{A}_k\}, r_k)$ on Y such that $\mathrm{cl}(V_\alpha) \subset U_\alpha$ for all k and $\alpha \in \mathcal{A}_k$. Define

$$t_k : \mathcal{A}_k \to \mathcal{T}(Y) \quad \text{and} \quad s_k : \mathcal{A}_k \to \mathcal{T}(Y)$$

by $t_k(\alpha) = U_\alpha$ and, respectively, $s_k(\alpha) = V_\alpha$. Also, define $p_k : \mathcal{A}_k \to \mathcal{P}(X)$ by $p_k(\alpha) = \theta^{-1}(\mathrm{cl}(s_k(\alpha)))$ for every $\alpha \in \mathcal{A}_k$. Then, $\langle (p_k, t_k; \mathcal{A}_k), r_k \rangle_{k\in\mathbb{N}}$ and $\langle (p_k, s_k; \mathcal{A}_k), r_k \rangle_{k\in\mathbb{N}}$ are as required. Indeed, each $\{p_k(\alpha)\colon\ \alpha \in \mathcal{A}_k\}$ is a locally finite closed cover of X because θ is usco. Next, for every $\alpha \in \mathcal{A}_k$,

$$p_k(\alpha) = \theta^{-1}\big(\mathrm{cl}\big(s_k(\alpha)\big)\big) \subset \Phi^{-1}\big(\mathrm{cl}\big(s_k(\alpha)\big)\big) \subset \Phi^{-1}\big(t_k(\alpha)\big)$$

because θ is a selection for Φ. That is, $(p_k, s_k; \mathcal{A}_k) \in \Theta(\Phi)$ and, respectively, $(p_k, t_k; \mathcal{A}_k) \in \Omega(\Phi)$. Obviously, for every k,

$$(p_{k+1}, s_{k+1}; \mathcal{A}_{k+1}) \ll_{r_k} (p_k, s_k; \mathcal{A}_k), \qquad (p_{k+1}, t_{k+1}; \mathcal{A}_{k+1}) \ll_{r_k} (p_k, t_k; \mathcal{A}_k),$$

and

$$\mathrm{mesh}(p_k, s_k; \mathcal{A}_k) \leqslant \mathrm{mesh}(p_k, t_k; \mathcal{A}_k) \leqslant \frac{1}{k}.$$

Take finally a point $x \in X$. Then $\mathrm{cl}(t_k(\alpha)) \subset B^d_{1/k}(\mathrm{cl}(s_k(\alpha)))$, $\alpha \in \mathcal{A}_k$, implies

$$\begin{aligned}\theta(x) &\subset \bigcup \big\{\mathrm{cl}\big(s_k(\alpha)\big)\colon\ \alpha \in \mathcal{A}_k \text{ and } x \in p_k(\alpha)\big\}\\ &\subset \bigcup \big\{\mathrm{cl}\big(t_k(\alpha)\big)\colon\ \alpha \in \mathcal{A}_k \text{ and } x \in p_k(\alpha)\big\}\\ &\subset \bigcup \big\{B^d_{1/k}\big(\mathrm{cl}\big(s_k(\alpha)\big)\big)\colon\ \alpha \in \mathcal{A}_k \text{ and } x \in p_k(\alpha)\big\} \subset B^d_{2/k}\big(\theta(x)\big),\end{aligned}$$

and therefore

$$\begin{aligned}\theta(x) &\subset \bigcap \big\{[p_k, s_k; \mathcal{A}_k](x)\colon\ k \in \mathbb{N}\big\}\\ &\subset \bigcap \big\{[p_k, t_k; \mathcal{A}_k](x)\colon\ k \in \mathbb{N}\big\} \subset \bigcap \big\{B^d_{2/k}(\theta(x))\colon\ k \in \mathbb{N}\} = \theta(x). \qquad \square\end{aligned}$$

For the proof of Theorem 2.3 we need also an indication for the "degree of compactness" of the subsets of Y (introduced by Kuratowski and used for approximations of compact-valued selections in [7]). Let $F \subset Y$. To every $S \subset Y$ and every $n \in \mathbb{N}$ we associate a number $\delta_n(S; F) \in [0, 1]$ by the formula:

$$\delta_n(S; F) = \inf\big(\{1\} \cup \{\varepsilon > 0\colon\ S \subset B^d_\varepsilon(S_0) \text{ for } S_0 \subset S \cap F \text{ with } |S_0| \leqslant n\}\big).$$

Also, we set $\delta_\omega(S; F) = \inf\{\delta_n(S; F)\colon\ n \in \mathbb{N}\}$.

Proof of Theorem 2.3. The implication (a) $\Rightarrow$ (b) follows from Lemma 2.4.

(b) $\Rightarrow$ (a) Let $\langle (p_k, t_k; \mathcal{A}_k), r_k \rangle_{k\in\mathbb{N}}$ be an $\Omega(\Phi)$-decomposition of θ. Note that, for every $x \in X$,

$$\lim_{k\to\infty} \delta_\omega\big([t_k, p_k; \mathcal{A}_k](x); \Phi(x)\big) \leqslant \lim_{k\to\infty} \mathrm{mesh}(p_k, t_k; \mathcal{A}_k) = 0$$

because $\{\mathrm{cl}(t_k(\alpha))\colon\ \alpha \in \mathcal{A}_k \text{ and } x \in p_k(\alpha)\}$ is a finite cover of $[t_k, p_k; \mathcal{A}_k](x)$ for every k. Also, note that

$$[t_{k+1}, p_{k+1}; \mathcal{A}_{k+1}](x) \subset [t_k, p_k; \mathcal{A}_k](x), \quad x \in X,$$

because $(p_{k+1}, t_{k+1}; \mathcal{A}_{k+1}) \ll_{r_k} (p_k, t_k; \mathcal{A}_k)$. Then, by (a) of [7, Lemma 3.2], each

$$\theta(x) = \bigcap \big\{[t_k, p_k; \mathcal{A}_k](x)\colon\ k \in \mathbb{N}\big\}$$

is a non-empty compact subset of $\Phi(x)$ because (Y,d) is complete. By (b) of the same lemma, θ is u.s.c. because so is each of the mappings $[t_k, p_k; \mathcal{A}_k]$. $\square$

We conclude this section with a refinement of Theorem 2.3 concerning finite-valued u.s.c. selection.

Theorem 2.5. *Let (Y,d) be a complete metric space, $\Phi : X \to \mathcal{F}(Y)$, $M \subset X$, and let $\theta : X \to 2^Y \cup \{\emptyset\}$ be a mapping which admits an $\Omega(\Phi)$-decomposition $\langle (p_k, t_k; \mathcal{A}_k), r_k\rangle_{k\in\mathbb{N}}$ such that $\mathrm{Ord}_M(p_k, t_k; \mathcal{A}_k) \leqslant m$ for some $m \in \mathbb{N}$ and every $k \geqslant k_0$. Then θ is an usco selection for Φ such that $|\theta(x)| \leqslant m$ for every $x \in M$.*

Proof. By virtue of Theorem 2.3, θ is an usco selection for Φ. Take a point $x \in M$ and then note that $\delta_m([p_k, t_k; \mathcal{A}_k](x); \Phi(x)) \leqslant \mathrm{mesh}(p_k, t_k; \mathcal{A}_k)$, $k \geqslant k_0$. Hence, $\lim_{k\to\infty} \delta_m(\theta_k(x); \Phi(x)) = 0$ and therefore, by [7, Lemma 3.2],

$$\left|\theta(x)\right| = \left|\bigcap\left\{[p_k, t_k; \mathcal{A}_k](x):\ k \in \mathbb{N}\right\}\right| \leqslant m. \qquad \square$$

3. Expansions of usco selections

Let (Y,d) be a metric space. For a mapping $\Phi : X \to \mathcal{F}(Y)$ we consider the following subset of $\Omega(\Phi)$:

$$\Upsilon(\Phi) = \left\{(p,t;\mathcal{A}) \in \Omega(\Phi):\ \left\{\mathrm{Int}(p(\alpha)):\ \alpha \in \mathcal{A}\right\} \text{ covers } X\right\}.$$

Definition 3.1. Suppose $A \subset X$ is closed and $\theta : A \to \mathcal{C}(Y)$ is a u.s.c. selection for $\Phi|A$. We shall say that $\psi : X \to 2^Y \cup \{\emptyset\}$ is an $\Upsilon(\Phi)$*-expansion* of θ if there exists an $\Upsilon(\Phi)$-decomposition $\langle (q_k, \ell_k; \mathcal{B}_k), \rho_k\rangle_{k\in\mathbb{N}}$ of ψ and a $\Theta(\Phi|A)$-decomposition $\langle (\eta_k, \lambda_k; \mathcal{B}_k), \rho_k\rangle_{k\in\mathbb{N}}$ of θ such that, for all k and $\beta \in \mathcal{B}_k$,

(a) $\mathrm{cl}(\lambda_k(\beta)) \subset \ell_k(\beta)$,
(b) $\eta_k(\beta) = \emptyset$ provided $q_k(\beta) \cap A = \emptyset$,
(c) $\eta_k(\beta) = \theta^{-1}(\mathrm{cl}(\lambda_k(\beta))) \subset \mathrm{Int}(q_k(\beta))$ provided $q_k(\beta) \cap A \neq \emptyset$.

First of all, let us observe that, by Theorem 2.3, every $\Upsilon(\Phi)$-expansion of θ has the following important property.

Proposition 3.2. *Let X be a topological space, (Y,d) be a complete metric space, $\Phi : X \to \mathcal{F}(Y)$, $A \subset X$ be closed, $\theta : A \to \mathcal{C}(Y)$ be a u.s.c. selection for $\Phi|A$, and let $\psi : X \to 2^Y \cup \{\emptyset\}$ be an $\Upsilon(\Phi)$-expansion of θ. Then ψ is an usco selection for Φ such that $\theta(x) \subset \psi(x)$ for every $x \in A$.*

In order to state the main result of this section, we consider also the following property of a mapping $\theta : A \to 2^Y$.

(3.3) Whenever $\{F_\alpha:\ \alpha \in \mathcal{A}\}$ is a locally finite family of closed subsets of Y, there exists a locally finite family $\{U_\alpha:\ \alpha \in \mathcal{A}\}$ of open subsets of X such that $\theta^{-1}(F_\alpha) \subset U_\alpha$ for every $\alpha \in \mathcal{A}$.

Theorem 3.4. *Let X be a normal space, $A \subset X$ closed, (Y, d) a metric space, $\Phi : X \to \mathcal{F}(Y)$ l.s.c. having the SFP, and let, for some $m \in \mathbb{N}$, $\theta : A \to \mathcal{C}_m(Y)$ be a u.s.c. selection for $\Phi|A$ satisfying* (3.3). *Then, there exists an $\Upsilon(\Phi)$-expansion ψ of θ such that $|\psi(x)| \leqslant m$ for every $x \in A$.*

Before turning to the proof of Theorem 3.4, let us recall that a set-valued mapping $\Phi : X \to \mathcal{F}(Y)$ has the *Selection Factorization Property*, or the SFP [17] (see also [3]), if for every $F \subset X$ closed and every locally finite (in Y) collection $\mathcal{U}$ of open subsets of Y such that $\Phi^{-1}(\mathcal{U}) = \{\Phi^{-1}(U)\colon U \in \mathcal{U}\}$ covers F there exists an open and locally finite (in F) cover of F which refines $\Phi^{-1}(\mathcal{U})$. Some of the most important examples of mappings Φ having the SFP and mappings θ satisfying (3.3) are given at the end of this section.

Proof of Theorem 3.4. By virtue of Lemma 2.4, θ has an $\Omega(\Phi|A)$-decomposition $\langle(p_k, t_k; \mathcal{A}_k), r_k\rangle_{k\in\mathbb{N}}$ and a $\Theta(\Phi|A)$-decomposition $\langle(p_k, s_k; \mathcal{A}_k), r_k\rangle_{k\in\mathbb{N}}$ such that, for every k and $\alpha \in \mathcal{A}_k$,

(1) $\mathrm{cl}(s_k(\alpha)) \subset t_k(\alpha)$ and $p_k(\alpha) = \theta^{-1}(\mathrm{cl}(s_k(\alpha)))$.

Note that each $\{\theta^{-1}(\mathrm{cl}(s_k(\alpha)))\colon \alpha \in \mathcal{A}_k\}$ is an index-refinement of the open family $\{\Phi^{-1}(t_k(\alpha))\colon \alpha \in \mathcal{A}_k\}$. Since X is normal and θ satisfies (3.3), we get a sequence $\{U(\alpha)\colon \alpha \in \mathcal{A}_k\}_{k\in\mathbb{N}}$ of locally finite open (in X) covers of A such that

$$p_k(\alpha) = \theta^{-1}\big(\mathrm{cl}\big(s_k(\alpha)\big)\big) \subset U(\alpha) \subset \mathrm{cl}\big(U(\alpha)\big) \subset \Phi^{-1}\big(t_k(\alpha)\big), \quad \alpha \in \mathcal{A}_k.$$

Inductively, taking in consideration that $(p_{k+1}, t_{k+1}; \mathcal{A}_{k+1}) \ll_{r_k} (p_k, t_k; \mathcal{A}_k)$ and using a result of Morita [16], we first modify the sequence $\{U(\alpha)\colon \alpha \in \mathcal{A}_k\}_{k\in\mathbb{N}}$ to a sequence $\{U_k(\alpha)\colon \alpha \in \mathcal{A}_k\}_{k\in\mathbb{N}}$ of locally finite open (in X) covers of A such that

(2) $p_k(\alpha) \subset U_k(\alpha) \subset U(\alpha),\ \alpha \in \mathcal{A}_k$,

(3) $U_{k+1}(\gamma) \subset U_k(r_k(\gamma)),\ \gamma \in \mathcal{A}_{k+1}$,

(4) $\bigcap\{\mathrm{cl}(U_k(\alpha))\colon \alpha \in \mathcal{B}\} \neq \emptyset$ implies $\bigcap\{p_k(\alpha)\colon \alpha \in \mathcal{B}\} \neq \emptyset$ for every non-empty finite $\mathcal{B} \subset \mathcal{A}_k$.

Let $\mathcal{A}_k^0, \mathcal{A}_k^1, \ldots, \mathcal{A}_k^k$ be $(k+1)$-disjoint copies of $\mathcal{A}_k$, say $\mathcal{A}_k^0 = \mathcal{A}_k$ and $\mathcal{A}_k^i = \{i\} \times \mathcal{A}_k,\ 1 \leqslant i \leqslant k$. Setting $\mathcal{B}_k = \mathcal{A}_k^0 \cup \mathcal{A}_k^1 \cup \cdots \cup \mathcal{A}_k^k$, we define as follows:

(5) $\ell_k : \mathcal{B}_k \to \mathcal{T}(Y)$ by $\ell_k|\mathcal{A}_k^0 = t_k$ and $\ell_k(i, \alpha) = t_k(\alpha)$ otherwise.

(6) $\rho_k : \mathcal{B}_{k+1} \to \mathcal{B}_k$ by $\rho_k|\mathcal{A}_k^0 = r_k$, $\rho_k(k+1, \gamma) = r_k(\gamma)$, and $\rho_k(i, \gamma) = (i, r_k(\gamma))$ otherwise.

(7) $\lambda_k : \mathcal{B}_k \to \mathcal{T}(Y)$ by $\lambda_k|\mathcal{A}_k^0 = s_k$ and $\lambda_k(i, \alpha) = s_k(\alpha)$ otherwise.

By induction on k, we shall now define maps $q_k : \mathcal{B}_k \to \mathcal{P}(X)$ such that

(i) $(q_k, \ell_k; \mathcal{B}_k) \in \Upsilon(\Phi)$,

(ii) $(q_{k+1}, \ell_{k+1}; \mathcal{B}_{k+1}) \ll_{\rho_k} (q_k, \ell_k; \mathcal{B}_k)$,

(iii) $q_k(\beta) \cap A = \emptyset$ provided $\beta \notin \mathcal{A}_k^0$,

(iv) $q_k(\beta) = \mathrm{cl}(U_k(\beta))$ provided $\beta \in \mathcal{A}_k^0$.

We first define $q_1 : \mathcal{B}_1 \to \mathcal{P}(X)$. For every $\beta \in \mathcal{A}_1$, following (iv), we let $q_1(\beta) = \mathrm{cl}(U_1(\beta))$. Since $U_1 = \bigcup\{U_1(\beta)\colon \beta \in \mathcal{A}_1^0\}$ is a neighbourhood of A, there is an open $X_1 \subset X$ such that $X \setminus U_1 \subset X_1 \subset \mathrm{cl}(X_1) \subset X \setminus A$. Then, let $\{V_\beta\colon \beta \in \mathcal{A}_1^1\}$ be a locally finite open (in $\mathrm{cl}(X_1)$) cover of $\mathrm{cl}(X_1)$ with $\mathrm{cl}(V_\beta) \subset \Phi^{-1}(\ell_1(\beta))$, $\beta \in \mathcal{A}_1^1$. Such $\{V_\beta\colon \beta \in \mathcal{A}_1^1\}$ certainly exists because Φ has the SFP and $\mathrm{cl}(X_1)$ is normal. Setting $q_1(\beta) = \mathrm{cl}(V_\beta)$, $\beta \in \mathcal{A}_1^1$, we finish the first step of our induction because $\mathcal{B}_1 = \mathcal{A}_1^0 \cup \mathcal{A}_1^1$.

Suppose now that, for some k, we have already defined the map $q_k : \mathcal{B}_k \to \mathcal{P}(X)$, and let us define $q_{k+1} : \mathcal{B}_{k+1} \to \mathcal{P}(X)$. We shall do that by defining each part $q_{k+1}|\mathcal{A}_{k+1}^i$ of the map q_{k+1} separately.

Definition 3.5 (of $q_{k+1}|\mathcal{A}_{k+1}^0$). Merely set $q_{k+1}(\gamma) = \mathrm{cl}(U_{k+1}(\gamma))$, $\gamma \in \mathcal{A}_{k+1}$.

Definition 3.6 (of $q_{k+1}|\mathcal{A}_{k+1}^{k+1}$). Note that, by (6), $\rho_k(\mathcal{A}_{k+1}^{k+1}) \subset \mathcal{A}_k^0$. Take an $\alpha \in \mathcal{A}_k^0$. Since

$$S_\alpha = q_k(\alpha) \setminus \bigcup \bigl\{U_{k+1}(\gamma)\colon \gamma \in \mathcal{A}_{k+1}^0\bigr\}$$

is a closed subset of $q_k(\alpha)$ with $S_\alpha \cap A = \emptyset$, there now is an open subset X_α of $q_k(\alpha)$ such that $S_\alpha \subset X_\alpha$ and $\mathrm{cl}(X_\alpha) \cap A = \emptyset$. Since $\mathcal{A}_{k+1}^{k+1}$ is a copy of $\mathcal{A}_{k+1}$, by (5) and (6),

$$\mathrm{cl}(X_\alpha) \subset q_k(\alpha) \subset \Phi^{-1}\bigl(\ell_k(\alpha)\bigr) = \bigcup \bigl\{\Phi^{-1}\bigl(\ell_{k+1}(\gamma)\bigr)\colon \gamma \in \rho_k^{-1}(\alpha) \cap \mathcal{A}_{k+1}^{k+1}\bigr\}.$$

Then, using the normality of $\mathrm{cl}(X_\alpha)$ and the SFP of Φ, we get a locally finite open (in $\mathrm{cl}(X_\alpha)$) cover $\{V_\gamma\colon \gamma \in \rho_k^{-1}(\alpha) \cap \mathcal{A}_{k+1}^{k+1}\}$ of $\mathrm{cl}(X_\alpha)$ such that

$$\mathrm{cl}(V_\gamma) \subset \Phi^{-1}\bigl(\ell_{k+1}(\gamma)\bigr).$$

Finally $q_{k+1}(\gamma) = \mathrm{cl}(V_\gamma)$, $\gamma \in \mathcal{A}_{k+1}^{k+1}$, completes the definition of $q_{k+1}|\mathcal{A}_{k+1}^{k+1}$.

Definition 3.7 (of $q_{k+1}|\mathcal{A}_{k+1}^i$, $1 \leqslant i \leqslant k$). Note that, by (6), $\rho_k^{-1}(\mathcal{A}_k^i) \subset \mathcal{A}_{k+1}^i$. Then, take an $\alpha \in \mathcal{A}_k^i$. It follows from the definitions of ℓ_k and ρ_k (see (5) and (6)) that

$$q_k(\alpha) \subset \Phi^{-1}\bigl(\ell_k(\alpha)\bigr) = \bigcup \bigl\{\Phi^{-1}\bigl(\ell_{k+1}(\gamma)\bigr)\colon \gamma \in \rho_k^{-1}(\alpha)\bigr\}.$$

In the same way as before, this implies the existence of a locally finite open (in $q_k(\alpha)$) cover $\{V_\gamma\colon \gamma \in \rho_k^{-1}(\alpha)\}$ of $q_k(\alpha)$ with $\mathrm{cl}(V_\gamma) \subset \Phi^{-1}(\ell_{k+1}(\gamma))$. Set finally $q_{k+1}(\gamma) = \mathrm{cl}(V_\gamma)$, $\gamma \in \mathcal{A}_{k+1}^i$.

Thus, the construction of $q_{k+1} : \mathcal{B}_{k+1} \to \mathcal{P}(Y)$ completes. We now finish the proof by showing that

$$\psi(x) = \bigcap \bigl\{[q_k, \ell_k; \mathcal{B}_k](x)\colon k \in \mathbb{N}\bigr\}, \qquad x \in X,$$

is the required one. Since, by (5),

$$\lim_{k\to\infty} (q_k, \ell_k; \mathcal{B}_k) = \lim_{k\to\infty} (p_k, t_k; \mathcal{A}_k) = 0,$$

it follows from (i) and (ii) that $\langle (q_k, \ell_k; \mathcal{B}_k), \rho_k\rangle_{k\in\mathbb{N}}$ is an $\Upsilon(\Phi)$-decomposition of ψ. For every k, define a map $\eta_k : \mathcal{B}_k \to \mathcal{P}(A)$ by $\eta_k(\beta) = \theta^{-1}(\mathrm{cl}(\lambda_k(\beta)))$ if $q_k(\beta) \cap A \neq \emptyset$ and $\eta_k(\beta) = \emptyset$ otherwise. In this way, by (1), (6) and (7), we get a $\Theta(\Phi|A)$-decomposition $\langle (\eta_k, \lambda_k; \mathcal{B}_k), \rho_k\rangle_{k\in\mathbb{N}}$ of θ. In fact, these decompositions define ψ as an $\Upsilon(\Phi)$-expansion of θ. Indeed, (a) of Definition 3.1 follows from (1) (see (5) and (7)); (b) follows from (iii); and (c) from (1), (2) and (iv).

Taking a point $x \in A$, it only remains to show that $|\psi(x)| \leqslant m$. To do that, we have, in effect, to show that $\delta_m(\psi(x); Y) = 0$. First, let us note that, by (iii) and (iv),

$$\mathcal{B}_k(x) = \{\beta \in \mathcal{B}_k \colon x \in q_k(\beta)\} = \{\beta \in \mathcal{A}_k \colon x \in \mathrm{cl}(U_k(\beta))\}$$

is a non-empty finite subset of $\mathcal{A}_k$ such that $\bigcap\{\mathrm{cl}(U_k(\beta))\colon \beta \in \mathcal{B}_k(x)\} \neq \emptyset$. Hence, by (4), there exists a point $x_k \in \bigcap\{p_k(\beta)\colon \beta \in \mathcal{B}_k(x)\}$. It finally follows from $|\theta(x_k)| \leqslant m$ and from the inclusions

$$\begin{aligned}\psi(x) &\subset \bigcup\{\mathrm{cl}(l_k(\beta))\colon \beta \in \mathcal{B}_k(x)\} = \bigcup\{\mathrm{cl}(t_k(\beta))\colon \beta \in \mathcal{B}_k(x)\}\\ &\subset \bigcup\{\mathrm{cl}(t_k(\alpha))\colon \alpha \in \mathcal{A}_k,\ t_k(\alpha) \cap \theta(x_k) \neq \emptyset\},\end{aligned}$$

that $\delta_m(\psi(x); Y) \leqslant \lim_{k\to\infty} \mathrm{mesh}(p_k, t_k; \mathcal{A}_k) = 0$. □

The rest of the section is devoted to some important examples. In what follows, we use $\mathcal{C}'(Y)$ to denote $\mathcal{C}(Y) \cup \{Y\}$.

Example 3.8 [17]. Let X be a normal space, Y a metrizable space with $w(Y) \leqslant \tau$, and let $\Phi : X \to \mathcal{F}(Y)$ be an l.s.c. mapping. Then Φ has the SFP provided X is τ-paracompact or X is τ-collectionwise normal and $\Phi(x) \in \mathcal{C}'(Y),\ x \in X$.

Our next examples are related to the property (3.3).

Example 3.9. Let X be a countably paracompact and τ-collectionwise normal space, $A \subset X$ be closed, and let Y be a space with $w(Y) \leqslant \tau$. Then every u.s.c. mapping $\theta : A \to \mathcal{C}(Y)$ satisfies (3.3).

Proof. Let $\{F_\alpha\colon \alpha \in \mathcal{A}\}$ be a locally finite family of closed subsets of Y. In order to check (3.3), we may suppose that each F_α is non-empty. Then $|\mathcal{A}| \leqslant \tau$ because $w(Y) \leqslant \tau$. Next, note that $\{\theta^{-1}(F_\alpha)\colon \alpha \in \mathcal{A}\}$ is a locally finite family of closed subsets of X because θ is usco and $A \subset X$ is closed. Since X is countably paracompact and τ-collectionwise normal, by a result of Dowker [6], this implies the existence of a locally finite family $\{U_\alpha\colon \alpha \in \mathcal{A}\}$ of open subsets of X such that $\theta^{-1}(F_\alpha) \subset U_\alpha$ for every $\alpha \in \mathcal{A}$. That completes the proof. □

Example 3.10. Let X be a τ-collectionwise normal space, $A \subset X$ be closed, Y be a finite-dimensional metrizable space with $w(Y) \leqslant \tau$, and let $m \in \mathbb{N}$. Then every u.s.c. mapping $\theta : A \to \mathcal{C}_m(Y)$ satisfies (3.3).

Proof. Suppose that $\{F_\alpha\colon \alpha \in \mathcal{A}\}$ is a locally finite family of non-empty closed subsets of Y, and let $a(y) = |\{\alpha \in \mathcal{A}\colon y \in F_\alpha\}|$ for every $y \in Y$. Then, setting $V_k = \{y \in Y\colon a(y) \leqslant k\}$, we get an open cover $\{V_k\colon k \in \mathbb{N}\}$ of Y. By assumption, Y is finite-dimensional, say $\dim(Y) \leqslant n$. Then, by [4], there exists a locally finite closed cover $\{Y_k\colon k \in \mathbb{N}\}$ of Y such that $Y_k \subset V_k,\ k \in \mathbb{N}$, and $\mathrm{Ord}(\{Y_k\colon k \in \mathbb{N}\}) \leqslant n + 1$. Set $M_k = \theta^{-1}(Y_k),\ k \in \mathbb{N}$, and then note that $\{M_k\colon k \in \mathbb{N}\}$ is a locally finite closed cover of A with $\mathrm{Ord}(\{M_k\colon k \in \mathbb{N}\}; A) \leqslant m(n + 1)$. Indeed, let $x \in A$. Then,

$$\big|\{k \in \mathbb{N}\colon x \in M_k\}\big| = \big|\{k \in \mathbb{N}\colon \theta(x) \cap Y_k \neq \emptyset\}\big| \leqslant m(n + 1)$$

because $|\theta(x)| \leqslant m$ and $|\{k \in \mathbb{N}\colon\ y \in Y_k\}| \leqslant (n+1)$ for every $y \in Y$. Since X is $\aleph_0$-collectionwise normal, by [9], this implies the existence of a locally finite open (in X) cover $\{W_k\colon\ k \in \mathbb{N}\}$ of A such that

(1) $M_k \subset W_k$ for every k.

Define set-valued mappings $\theta_k : M_k \to \mathcal{C}(Y_k)$ by $\theta_k(x) = \theta(x) \cap Y_k$, $x \in M_k$. Note that each θ_k is u.s.c. because $Y_k \subset Y$ is closed. Also, note that

(2) $\theta^{-1}(F_\alpha) = \bigcup\{\theta_k^{-1}(F_\alpha)\colon\ k \in \mathbb{N}\}$, $\alpha \in \mathcal{A}$,

because $\{Y_k\colon\ k \in \mathbb{N}\}$ covers Y. Let us now check that, for every k,

(3) $\mathrm{Ord}(\{\theta_k^{-1}(F_\alpha)\colon\ \alpha \in \mathcal{A}\}; M_k) < km$.

Indeed, take a point $x \in M_k$. Then,

$$\begin{aligned}\left|\left\{\alpha \in \mathcal{A}\colon\ x \in \theta_k^{-1}(F_\alpha)\right\}\right| &= \left|\left\{\alpha \in \mathcal{A}\colon\ x \in \theta_k(x) \cap F_\alpha \neq \emptyset\right\}\right| \\ &= \sum\left\{a(y)\colon\ y \in \theta_k(x)\right\} \leqslant mk\end{aligned}$$

because $|\theta_k(x)| \leqslant m$ and $\theta_k(x) \subset V_k$.

The proof now completes as follows. Because of the τ-collectionwise normality of X, by (3) and [9], for every k there is a locally finite open (in X) family $\{U_\alpha^k\colon\ \alpha \in \mathcal{A}\}$ such that $\theta_k^{-1}(F_\alpha) \subset U_\alpha^k$. Set

$$U_\alpha = \bigcup\left\{U_\alpha^k \cap W_k\colon\ k \in \mathbb{N}\right\}.$$

By (1) and (2), this $\{U_\alpha\colon\ \alpha \in \mathcal{A}\}$ satisfies all our requirements. $\square$

Example 3.11. Let X be a τ-collectionwise normal space, $A \subset X$ be closed, (Y, d) be a complete metric space with $w(Y) \leqslant \tau$, and let $\phi : A \to \mathcal{C}(Y)$ be an l.s.c. mapping. Then every u.s.c. selection $\theta : A \to \mathcal{C}(Y)$ for ϕ satisfies (3.3).

Proof. Let $H(d)$ be the *Hausdorff distance* on $\mathcal{C}(Y)$ generated by d. To recall that, for $K, Q \in \mathcal{C}(Y)$,

$$H(d)(K, Q) = \inf\left\{\varepsilon > 0\colon\ K \subset B_\varepsilon^d(Q) \ \text{ and } \ Q \subset B_\varepsilon^d(K)\right\}.$$

As is well known, $(\mathcal{C}(Y), H(d))$ is a complete metric space with $w(\mathcal{C}(Y)) \leqslant \tau$ because so is (Y, d). In what follows, we consider $\mathcal{C}(Y)$ as a topological space with the topology generated by the metric $H(d)$. Let now $\theta : A \to \mathcal{C}(Y)$ be a u.s.c. selection for ϕ. We define a set-valued mapping $\mathcal{C}_\theta(\phi) : A \to 2^{\mathcal{C}(Y)}$ by the formula:

$$\mathcal{C}_\theta(\phi)(x) = \left\{K \in \mathcal{C}(Y)\colon\ \theta(x) \subset K \subset \phi(x)\right\}, \quad x \in A.$$

Clearly, each $\mathcal{C}_\theta(\phi)(x)$ is a non-empty compact subset of $\mathcal{C}(Y)$. We now show that $\mathcal{C}_\theta(\phi)$ is l.s.c. Let $\varepsilon > 0$, $x \in A$ and let $K \in \mathcal{C}_\theta(\phi)(x)$. Since ϕ is l.s.c. and $K \subset \phi(x)$ is compact, it follows from [13, Lemma 11.3] that

$$V = \left\{z \in A\colon\ K \subset B_\varepsilon^d(\phi(z))\right\}$$

is a neighbourhood of x in A. On the other hand,

$$W = \left\{z \in A\colon\ \theta(z) \subset B_\varepsilon^d(K)\right\}$$

also defines a neighbourhood of x in A because θ is u.s.c. and $\theta(x) \subset B_\varepsilon^d(K)$. Set $U = W \cap V$. Since K is compact, $z \in U$ implies the existence of a compact $P \subset \phi(z)$ with $H(d)(P, K) < \varepsilon$. Then, setting $Q = P \cup \theta(z)$, we get a point $Q \in \mathcal{C}_\theta(\phi)(z)$ such that $H(d)(Q, K) < \varepsilon$. That is, $\mathcal{C}_\theta(\phi)$ is l.s.c.

Define now a set-valued mapping $\Lambda : X \to \mathcal{C}'(\mathcal{C}(Y))$ by $\Lambda(x) = \mathcal{C}_\theta(\phi)(x)$ if $x \in A$ and $\Lambda(x) = \mathcal{C}(Y)$ otherwise. Note that Λ is l.s.c. because $\mathcal{C}_\theta(\phi)$ is l.s.c. and $A \subset X$ is closed (see [12]). Then, by [17, Theorem 4.4], there exist a u.s.c. $\Psi : X \to \mathcal{C}(\mathcal{C}(Y))$ and an l.s.c. $\Phi : X \to \mathcal{C}(\mathcal{C}(Y))$ such that $\Phi(x) \subset \Psi(x) \subset \Lambda(x)$ for every $x \in X$. Set

$$\varphi = \bigcup \Phi(x) \quad \text{and} \quad \psi(x) = \bigcup \Psi(x), \quad x \in X.$$

In this way, by [11, Theorem 2.5], we get two compact-valued mappings $\varphi : X \to \mathcal{C}(Y)$ and $\psi : X \to \mathcal{C}(Y)$. Note that

(1) $\varphi(x) \subset \psi(x),\ x \in X$,

(2) $\theta(x) \subset \varphi(x),\ x \in A$.

Also, note that φ is l.s.c. and ψ is u.s.c. Indeed, take a point $x \in X$, and let $\varepsilon > 0$. The lower semi-continuity of φ follows from the fact that Φ is l.s.c. and that

$$x \in \left\{z \in X\colon\ \Phi(x) \subset B_\varepsilon^{H(d)}(\Phi(z))\right\} \subset \left\{z \in X\colon\ \varphi(x) \subset B_\varepsilon^d(\varphi(z))\right\}.$$

In the same way, ψ is u.s.c. because so is Ψ and because

$$x \in \left\{z \in X\colon\ \Psi(z) \subset B_\varepsilon^{H(d)}(\Psi(x))\right\} \subset \left\{z \in X\colon\ \psi(z) \subset B_\varepsilon^d(\psi(x))\right\}.$$

We now accomplish the proof as follows. Let $\{F_\alpha\colon\ \alpha \in \mathcal{A}\}$ be a locally finite family of closed subsets of Y. Since Y is metrizable, there exists a locally finite family $\{V_\alpha\colon\ \alpha \in \mathcal{A}\}$ of open subsets of Y such that $F_\alpha \subset V_\alpha,\ \alpha \in \mathcal{A}$. Since φ is l.s.c., $U_\alpha = \varphi^{-1}(V_\alpha),\ \alpha \in \mathcal{A}$, defines a family of open subset of X. By (1), $\{U_\alpha\colon\ \alpha \in \mathcal{A}\}$ is locally finite because, by the upper semi-continuity of ψ, so is $\{\psi^{-1}(\mathrm{cl}(V_\alpha))\colon\ \alpha \in \mathcal{A}\}$. Finally, by (2), $\theta^{-1}(F_\alpha) \subset U_\alpha,\ \alpha \in \mathcal{A}$. $\square$

4. A dimension-type improvement

The present section is devoted to the following improvement of Theorem 3.4.

Theorem 4.1. *Let X be a normal space, $A \subset X$ be closed with $\dim_X(X \setminus A) \leqslant n$, (Y, d) be a complete metric space, $\Phi : X \to \mathcal{F}(Y)$ be l.s.c. having the SFP, and let, for some $m \leqslant n + 1$, $\theta : A \to \mathcal{C}_m(Y)$ be a u.s.c. selection for $\Phi|A$ satisfying (3.3). Then θ admits an $\Upsilon(\Phi)$-expansion $\varphi : X \to \mathcal{C}_{n+1}(Y)$ such that $|\varphi(x)| \leqslant m$ for every $x \in A$.*

To prepare for the proof of Theorem 4.1, we need the following lemmas.

Lemma 4.2. *Let (Y, d) be a metric space, X be a normal space, $A \subset X$ be closed, and let $\psi : X \to \mathcal{C}(Y)$ be u.s.c. such that, for some $m \in \mathbb{N}$, $|\psi(x)| \leqslant m$ for every $x \in A$. Then, there exists a closed G_δ-subset B of X such that $A \subset B$ and $|\psi(x)| \leqslant m$ for every $x \in B$.*

Proof. Since X is normal and $A \subset X$ is closed, it suffices to show that there exists a G_δ-subset D of X such that $A \subset D$ and $|\psi(x)| \leqslant m$ for all $x \in D$. Towards this end, let $V_k = \bigcup\{V_k(x)\colon x \in A\}$, where

$$V_k(x) = \{z \in X\colon \psi(z) \subset B^d_{1/k}(\psi(x))\},$$

and let us check that $D = \bigcap\{V_k\colon k \in \mathbb{N}\}$ is as required. Since ψ is u.s.c., D is a G_δ-subset of X. Take a point $x \in D$. For every k, there exists a point $x_k \in A$ with $x \in V_k(x_k)$. Note that $|\psi(x_k)| \leqslant m$ and $\psi(x) \subset B^d_{1/k}(\psi(x_k))$. So, by definition, $\delta_m(\psi(x); Y) \leqslant 1/k$. That is, $\delta_m(\psi(x); Y) = 0$ and therefore $|\psi(x)| \leqslant m$. □

Lemma 4.3. *Let Z be a normal space, $F \subset Z$ be closed with $\dim(F) \leqslant n$, (Y, d) be a metric space, $\Phi : Z \to \mathcal{F}(Y)$ have the SFP, and let $(q, \ell; \mathcal{B}) \in \Upsilon(\Phi)$. Then for every locally finite open cover $\mathcal{W}$ of Y there exists a $(p, t; \mathcal{A}) \in \Upsilon(\Phi)$ and a map $r : \mathcal{A} \to \mathcal{B}$ such that*

(i) *$t(\mathcal{A})$ refines $\mathcal{W}$,*
(ii) *$(p, t; \mathcal{A}) \ll_r (q, \ell; \mathcal{B})$,*
(iii) *$\mathrm{Ord}_F(p, t; \mathcal{A}) \leqslant n + 1$.*

Proof. Set $\mathcal{A} = \mathcal{B} \times \mathcal{W}$, and let $r : \mathcal{A} \to \mathcal{B}$ and $w : \mathcal{A} \to \mathcal{W}$ be the projections. Define a map $t : \mathcal{A} \to \mathcal{T}(Y)$ by $t(\beta, W) = \ell(\beta) \cap W$. Thus, we get a locally finite open cover $\{t(\alpha)\colon \alpha \in \mathcal{A}\}$ of Y which refines $\mathcal{W}$. Take a $\beta \in \mathcal{B}$, and then note that $\{\Phi^{-1}(t(\alpha))\colon \alpha \in r^{-1}(\beta)\}$ covers $q(\beta)$ because $\ell(\beta) = \bigcup\{t(\alpha)\colon \alpha \in r^{-1}(\beta)\}$. Therefore, since Φ has the SFP and $q(\beta) \subset Z$ is closed, there exists a locally finite open (in $q(\beta)$) cover $\{S_\alpha\colon \alpha \in r^{-1}(\beta)\}$ of $q(\beta)$ such that each S_α is a subset of $\Phi^{-1}(t(\alpha))$. Then, setting

$$V_\alpha = S_\alpha \cap \mathrm{Int}\big(q(r(\alpha))\big),$$

we get a locally finite open cover $\{V_\alpha\colon \alpha \in \mathcal{A}\}$ of X because

$$X = \bigcup \big\{\mathrm{Int}\big(q(r(\alpha))\big)\colon \alpha \in \mathcal{A}\big\}.$$

Since now $\dim(F) \leqslant n$ and Z is normal, by [4], $\{V_\alpha\colon \alpha \in \mathcal{A}\}$ admits an open index-refinement $\{U_\alpha\colon \alpha \in \mathcal{A}\}$ with $\mathrm{Ord}(\{U_\alpha\colon \alpha \in \mathcal{A}\}; F) \leqslant n + 1$. By a Lefshetz's lemma [10], there exists an open cover $\{P_\alpha\colon \alpha \in \mathcal{A}\}$ of Z such that $\mathrm{cl}(P_\alpha) \subset U_\alpha$, $\alpha \in \mathcal{A}$. Define finally $p : \mathcal{A} \to \mathcal{P}(Z)$ by $p(\alpha) = \mathrm{cl}(P_\alpha)$, $\alpha \in \mathcal{A}$. These $(p, t; \mathcal{A}) \in \Upsilon(\Phi)$ and $r : \mathcal{A} \to \mathcal{B}$ satisfy all our requirements. □

Proof of Theorem 4.1. Let X, (Y, d), $A \subset X$, $\Phi : X \to \mathcal{F}(Y)$ and $\theta : A \to \mathcal{C}_m(Y)$ be as in that theorem. By Theorem 3.4, θ admits an $\Upsilon(\Phi)$-expansion ψ such that $|\psi(x)| \leqslant m$ for every $x \in A$. By definition, there exists an $\Upsilon(\Phi)$-decomposition $\langle(q_k, \ell_k; \mathcal{B}_k), \rho_k\rangle_{k\in\mathbb{N}}$ of ψ and a $\Theta(\Phi|A)$-decomposition $\langle(\eta_k, \lambda_k; \mathcal{B}_k), \rho_k\rangle_{k\in\mathbb{N}}$ of θ such that, for all k and $\beta \in \mathcal{B}_k$,

(a) $\mathrm{cl}(\lambda_k(\beta)) \subset \ell_k(\beta)$,
(b) $\eta_k(\beta) = \emptyset$ provided $q_k(\beta) \cap A = \emptyset$,
(c) $\eta_k(\beta) = \theta^{-1}(\mathrm{cl}(\lambda_k(\beta))) \subset \mathrm{Int}(q_k(\beta))$ provided $q_k(\beta) \cap A \neq \emptyset$.

By Proposition 3.2, ψ is an usco selection for Φ. Therefore, by Lemma 4.2, there exists a closed G_δ-subset $B \subset X$ such that $A \subset B$ and $|\psi(x)| \leqslant m$ for all $x \in B$. Since X is normal, $B = \bigcap\{G_k\colon\ k \in \mathbb{N}\}$ where each $G_k \subset X$ is open and $\mathrm{cl}(G_{k+1}) \subset G_k$. Note that $\dim(X\backslash G_k) \leqslant n$ because $X\backslash G_k \subset X\backslash A$.

For technical reasons only, we now set $\mathcal{A}_0 = \mathcal{B}_0 = \{0\}$, $p_0(0) = q_0(0) = X$ and $t_0(0) = \ell_0(0) = Y$. Define $\rho_0 : \mathcal{B}_1 \to \mathcal{B}_0$ by $\rho_0(\beta) = 0$, $\beta \in \mathcal{B}_1$. Also, set $G_0 = X$. Since X is normal and $\mathrm{cl}(G_{k+1}) \subset G_k$, there are open sets $V_k, U_k \subset X$ such that

$$\mathrm{cl}(G_{k+1}) \subset V_k \subset \mathrm{cl}(V_k) \subset U_k \subset \mathrm{cl}(U_k) \subset G_k.$$

By induction, we shall construct a sequence $\{(p_k, t_k; \mathcal{A}_k)\colon\ k = 0, 1, \ldots\}$ in $\Upsilon(\Phi)$ and maps $r_k : \mathcal{A}_{k+1} \to \mathcal{A}_k$ such that, for every k,

(1) $\mathcal{B}_k \subset \mathcal{A}_k$,
(2) $t_k|\mathcal{B}_k = \ell_k$ and $t_k(\mathcal{A}_k\backslash\mathcal{B}_k)$ refines $\ell_k(\mathcal{B}_k)$,
(3) $p_k(\alpha) = q_k(\alpha) \cap \mathrm{cl}(U_k)$, $\alpha \in \mathcal{B}_k$,
(4) $p_k(\alpha) \cap G_{k+1} = \emptyset$, $\alpha \in \mathcal{A}_k\backslash\mathcal{B}_k$,
(5) $\mathrm{Ord}_{X\backslash G_k}(p_k, t_k; \mathcal{A}_k) \leqslant n + 1$,
(6) $r_k|\mathcal{B}_{k+1} = \rho_k$ and $r_k^{-1}(\mathcal{A}_k\backslash\mathcal{B}_k) \subset \mathcal{A}_{k+1}\backslash\mathcal{B}_{k+1}$,
(7) $(p_{k+1}, t_{k+1}; \mathcal{A}_{k+1}) \ll_{r_k} (p_k, t_k; \mathcal{A}_k)$.

Since $(p_0, t_0; \mathcal{A}_0)$ was defined above, we may suppose that $(p_{k-1}, t_{k-1}; \mathcal{A}_{k-1})$ is already defined, and we must define $(p_k, t_k; \mathcal{A}_k)$ and r_{k-1}. Towards this end, set $Z = X\backslash V_k$ and then define $h : \mathcal{A}_{k-1} \to \mathcal{P}(Z)$ by $h(\alpha) = p_{k-1}(\alpha) \cap Z$. Since $(h, t_{k-1}; \mathcal{A}_{k-1}) \in \Upsilon(\Phi|Z)$, by Lemma 4.3 applied to $\mathcal{W} = \ell_k(\mathcal{B}_k)$ and $F = X\backslash G_k$, there is $(p, t; \mathcal{A}) \in \Upsilon(\Phi|Z)$ and a map $r : \mathcal{A} \to \mathcal{B}_k$ such that

(i) $t(\mathcal{A})$ refines $\ell_k(\mathcal{B}_k)$,
(ii) $(p, t; \mathcal{A}) \ll_r (h, t_{k-1}; \mathcal{A}_{k-1})$,
(iii) $\mathrm{Ord}_{X\backslash G_k}(p, t; \mathcal{A}) \leqslant n + 1$.

Assuming $\mathcal{A} \cap \mathcal{B}_k = \emptyset$, we now let $\mathcal{A}_k = \mathcal{A} \cup \mathcal{B}_k$. Next, we define as follows: a map $t_k : \mathcal{A}_k \to \mathcal{T}(Y)$ by $t_k|\mathcal{A} = t$ and $t_k|\mathcal{B}_k = \ell_k$; a map $p_k : \mathcal{A}_k \to \mathcal{P}(X)$ by $p_k|\mathcal{A} = p$ and $p_k(\beta) = q_k(\beta) \cap \mathrm{cl}(U_k)$, $\beta \in \mathcal{B}_k$; and a map $r_{k-1} : \mathcal{A}_k \to \mathcal{A}_{k-1}$ by $r_{k-1}|\mathcal{B}_{k-1} = \rho_{k-1}$ and $r_{k-1}|\mathcal{A} = r$. Let us check that these $(p_k, t_k; \mathcal{A}_k) \in \Upsilon(\Phi)$ and r_{k-1} are as required. By construction, (1), (3), (4) and (6) hold true. Next, (2) follows from (i), and (5) from (iii). It only remains to check (7). It follows from (ii) and the properties of ρ_{k-1} that

$$t_{k-1}(\alpha) = \bigcup\bigl\{t_k(\gamma)\colon\ \gamma \in r_{k-1}^{-1}(\alpha)\bigr\}, \quad \alpha \in \mathcal{A}_{k-1}.$$

Take a $\gamma \in \mathcal{A}_k$. To show finally that $p_k(\gamma) \subset p_{k-1}(r_{k_1}(\gamma))$, we distinguish two case: If $\gamma \in \mathcal{A}$, then $r_{k-1}(\gamma) = r(\gamma)$ and therefore, by (ii),

$$p_k(\gamma) = p(\gamma) \subset h\bigl(r(\gamma)\bigr) \subset p_{k-1}\bigl(r(\gamma)\bigr) = p_{k-1}\bigl(r_{k-1}(\gamma)\bigr).$$

If $\gamma \in \mathcal{B}_k$, then $r_{k-1}(\gamma) = \rho_{k-1}(\gamma)$ and therefore

$$p_k(\gamma) = q_k(\gamma) \cap \mathrm{cl}(U_k) \subset q_{k-1}\bigl(\rho_{k-1}(\gamma)\bigr) \cap \mathrm{cl}(U_{k-1}) = p_{k-1}\bigl(r_{k-1}(\gamma)\bigr).$$

This completes our inductive construction.

We now finish the proof showing that

$$\varphi(x) = \bigcap\bigl\{[p_k, t_k; \mathcal{A}_k](x)\colon\ k \in \mathbb{N}\bigr\}, \quad x \in X,$$

satisfies all our requirements. Towards this end, we first note that, by (2),

$$\lim_{k\to\infty} \operatorname{mesh}(p_k, t_k; \mathcal{A}_k) \leqslant \lim_{k\to\infty} \operatorname{mesh}(q_k, \ell_k; \mathcal{B}_k) = 0.$$

Therefore, by (7), $\langle(p_k, t_k; \mathcal{A}_k), r_k\rangle_{k\in\mathbb{N}}$ is an $\Upsilon(\Phi)$-decomposition of φ. Next, for every k, define a map $\xi_k : \mathcal{A}_k \to \mathcal{T}(Y)$ by $\xi_k(\alpha) = \lambda_k(\alpha)$ if $\alpha \in \mathcal{B}_k$ and $\xi(\alpha) = \emptyset$ otherwise. Also, define maps $\pi_k : \mathcal{A}_k \to \mathcal{P}(X)$ by $\pi_k(\alpha) = \eta(\alpha)$ if $\alpha \in \mathcal{B}_k$ and $\pi_k(\alpha) = \emptyset$ otherwise. In this way, by (6), we get a $\Theta(\Phi|A)$-decomposition $\langle(\pi_k, \xi_k; \mathcal{A}_k), r_k\rangle_{k\in\mathbb{N}}$ of θ because so is $\langle(\eta_k, \lambda_k; \mathcal{B}_k), \rho_k\rangle_{k\in\mathbb{N}}$. In effect, this defines φ as an $\Upsilon(\Phi)$-expansion of θ. Indeed, by (a), (1) and (2), $\operatorname{cl}(\xi_k(\alpha)) \subset t_k(\alpha)$, $\alpha \in \mathcal{A}_k$. Next, by (b), (3) and (4), $p_k(\alpha) \cap A = \emptyset$ implies $\pi_k(\alpha) = \emptyset$. Finally, by (3) and (4), $p_k(\alpha) \cap A \neq \emptyset$ implies $\alpha \in \mathcal{B}_k$ and therefore, by (c),

$$\pi_k(\alpha) = \theta^{-1}\big(\operatorname{cl}\big(\xi_k(\alpha)\big)\big) \subset \operatorname{Int}\big(q_k(\alpha)\big) \cap U_k \subset \operatorname{Int}\big(p_k(\alpha)\big).$$

Take now a point $x \in X$. In case $x \in B$, it follows from (3) and (4) that $\varphi(x) = \psi(x)$. Hence, $|\varphi(x)| = |\psi(x)| \leqslant m \leqslant n+1$. In case $x \notin B$, it follows from (5) and Theorem 2.5 that $|\varphi(x)| \leqslant n+1$. □

5. Controlled u.s.c. "extensions" of u.s.c. selections

In this section we first prove Theorems 1.6 and 1.7 from the introduction. Since every paracompact space is normal and τ-paracompact (for all τ), these theorems are consequences of the following two more general results.

Theorem 5.1. *Let X be a τ-paracompact normal space, A a closed subset of X, Y a completely metrizable space with $w(Y) \leqslant \tau$, $\Phi : X \to \mathcal{F}(Y)$ l.s.c., and let, for some $m \in \mathbb{N}$, $\theta : A \to \mathcal{C}_m(Y)$ be a u.s.c. selection for $\Phi|A$. Then Φ admits a u.s.c. selection $\varphi : X \to \mathcal{C}(Y)$ such that $\theta(x) \subset \varphi(x)$ and $|\varphi(x)| \leqslant m$ for all $x \in A$.*

Proof. By Example 3.5, Φ has the SFP and, by Example 3.6, θ satisfies (3.3). Then, Proposition 3.2 and Theorem 3.4 complete the proof. □

Theorem 5.2. *Let X be a τ-paracompact normal space, A a closed subset of X with $\dim_X(X\setminus A) \leqslant n$, Y a completely metrizable space with $w(Y) \leqslant \tau$, $\Phi : X \to \mathcal{F}(Y)$ l.s.c., and let, for some $m \leqslant n+1$, $\theta : A \to \mathcal{C}_m(Y)$ be a u.s.c. selection for $\Phi|A$. Then Φ admits a u.s.c. selection $\varphi : X \to \mathcal{C}_{n+1}(Y)$ such that $\theta(x) \subset \varphi(x)$ and $|\varphi(x)| \leqslant m$ for all $x \in A$.*

Proof. Repeat precisely the previous proof but now use Theorem 4.1 instead of Theorem 3.4. □

The rest of the section is devoted to collectionwise normal versions of Theorems 5.1 and 5.2.

Theorem 5.3. *Let X be a τ-collectionwise normal space, A a closed subset of X, Y a completely metrizable space with $w(Y) \leqslant \tau$, $\Phi : X \to \mathcal{C}'(Y)$ l.s.c., and let, for some*

$m \in \mathbb{N}$, $\theta : A \to \mathcal{C}_m(Y)$ be a u.s.c. selection for $\Phi|A$. Then each of the following implies the existence of a u.s.c. selection $\varphi : X \to \mathcal{C}(Y)$ for Φ such that $\theta(x) \subset \varphi(x)$ and $|\varphi(x)| \leqslant m$ for all $x \in A$.

(a) *X is countably paracompact.*
(b) *Y is finite-dimensional.*
(c) *$\Phi(x)$ is compact for every $x \in A$.*

Proof. Since Φ has the SFP (Example 3.6), following the proof of Theorem 5.1, it suffices to show that θ satisfies (3.3). That this is so, it follows from Example 3.7 for the case of (a); from Example 3.8 for the case of (b); and from Example 3.9 for the case of (c). □

Theorem 5.4. *Let X be a τ-collectionwise normal space, A a closed subset of X with $\dim_X(X \setminus A) \leqslant n$, Y a completely metrizable space with $w(Y) \leqslant \tau$, $\Phi : X \to \mathcal{C}'(Y)$ l.s.c., and let, for some $m \leqslant n+1$, $\theta : A \to \mathcal{C}_m(Y)$ be a u.s.c. selection for $\Phi|A$. Then each of the following implies the existence of a u.s.c. selection $\varphi : X \to \mathcal{C}_{n+1}(Y)$ for Φ such that $\theta(x) \subset \varphi(x)$ and $|\varphi(x)| \leqslant m$ for all $x \in A$.*

(a) *X is countably paracompact.*
(b) *Y is finite-dimensional.*
(c) *$\Phi(x)$ is compact for every $x \in A$.*

Proof. Since Φ has the SFP and θ satisfies (3.3), this follows from Theorem 4.1 and Proposition 3.2. □

6. Some applications for the covering dimension of normal spaces

In this section we apply the results of the previous section to obtain some characterizations of the covering dimension of normal spaces in terms of “extensions” of usco mappings. In what follows, we use Y_τ to denote the discrete space of the cardinality τ.

Theorem 6.1. *For a normal space X and $n \geqslant 0$, the following conditions are equivalent*:

(a) $\dim(X) \leqslant n$.
(b) *Whenever $A \subset X$ is closed and Y is a compact metric space, every u.s.c. $\theta : A \to \mathcal{C}_{n+1}(Y)$ is a selection for $\varphi|A$ for some u.s.c. $\varphi : X \to \mathcal{C}_{n+1}(Y)$.*
(c) *Whenever $A \subset X$ is closed, every u.s.c. $\theta : A \to \mathcal{C}_{n+1}(Y_{n+2})$ is a selection for $\varphi|A$ for some u.s.c. $\varphi : X \to \mathcal{C}_{n+1}(Y_{n+2})$.*

Proof. (a)$\Rightarrow$(b) follows from Theorem 5.4(c) with $\Phi(x) = Y$, $x \in X$. (b)$\Rightarrow$(c) is obvious. As for finally (c)$\Rightarrow$(a), suppose that $\{F_y\colon\ y \in Y_{n+2}\}$ is a family of closed subset of X with $\bigcap\{F_y\colon\ y \in Y_{n+2}\} = \emptyset$. Set $A = \bigcup\{F_y\colon\ y \in Y_{n+2}\}$. Next, define a u.s.c. mapping $\theta : A \to \mathcal{C}_{n+1}(Y_{n+2})$ by $\theta(x) = \{y \in Y_{n+2}\colon\ x \in F_y\}$, $x \in A$. By (c), θ is a selection for $\varphi|A$ for some u.s.c. $\varphi : X \to \mathcal{C}_{n+1}(Y_{n+2})$. Then, setting $G_y = \varphi^{-1}(\{y\})$, $y \in Y_{n+2}$, we get a closed cover $\{G_y\colon\ y \in Y_{n+2}\}$ of X such that each F_y is a subset of G_y and $\bigcap\{G_y\colon\ y \in Y_{n+2}\} = \emptyset$. Hence, $\dim(X) \leqslant n$. □

Our next result is a characterization of the relative dimension of open subsets of normal spaces.

Theorem 6.2. *For a normal space X, a closed $B \subset X$ and $n \geqslant 0$, the following conditions are equivalent*:

(a) $\dim_X(X \setminus B) \leqslant n$.

(b) *If $A \subset X$ is a closed subset containing B and Y is a compact metric space, then every u.s.c. $\theta : A \to \mathcal{C}_{n+1}(Y)$ is a selection for $\varphi|A$ for some u.s.c. $\varphi : X \to \mathcal{C}_{n+1}(Y)$.*

(c) *If $A \subset X$ is a closed subset containing B, then every u.s.c. $\theta : A \to \mathcal{C}_{n+1}(Y_{n+2})$ is a selection for $\varphi|A$ for some u.s.c. $\varphi : X \to \mathcal{C}_{n+1}(Y_{n+2})$.*

Proof. (a) $\Rightarrow$ (b) follows from Theorem 5.4(c) because $\dim_X(X \setminus A) \leqslant n$ for every $A \supset B$. (b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a) Taking an $S \subset X \setminus B$ which is closed in X, we have to show that $\dim(S) \leqslant n$. To see that this so, we shall use Theorem 6.1. Namely, let $F \subset S$ be closed and let $\psi : F \to \mathcal{C}_{n+1}(Y_{n+2})$ be u.s.c. Set $A = F \cup B$. Next, pick a fixed $y \in Y_{n+2}$ and then define a u.s.c. $\theta : A \to \mathcal{C}_{n+1}(Y_{n+2})$ by $\theta(x) = \psi(x)$ if $x \in F$ and $\theta(x) = \{y\}$ otherwise. By (c), θ is a selection for $\varphi|A$ for some u.s.c. $\varphi : X \to \mathcal{C}_{n+1}(Y_{n+2})$. Then $\varphi|S : S \to \mathcal{C}_{n+1}(Y_{n+2})$ is u.s.c. such that $\psi(x) \subset \varphi(x)$ for all $x \in F$. So, by virtue of Theorem 6.1, $\dim(S) \leqslant n$. □

Corollary 6.3 (Dowker [5]). *Let X be a normal space and let, for some closed $B \subset X$, $\dim(B) \leqslant n$ and $\dim_X(X \setminus B) \leqslant n$. Then $\dim(X) \leqslant n$.*

Proof. In order to use Theorem 6.1, let $A \subset X$ be closed, Y be a compact metric space, and let $\theta : A \to \mathcal{C}_{n+1}(Y)$ be u.s.c. Since $\dim(B) \leqslant n$, by Theorem 6.1, there exists a u.s.c. mapping $\psi : B \to \mathcal{C}_{n+1}(Y)$ such that $\theta(x) \subset \psi(x)$ for every $x \in A \cap B$. Set $A_0 = A \cup B$, and then define a u.s.c. $\theta_0 : A_0 \to \mathcal{C}_{n+1}(Y)$ by $\theta_0(x) = \psi(x)$ if $x \in B$ and $\theta_0(x) = \theta(x)$ otherwise. By Theorem 6.2, θ_0 is a selection for $\varphi|A_0$ for some u.s.c. $\varphi : X \to \mathcal{C}_{n+1}(Y)$. In particular, $\theta(x) \subset \varphi(x)$ for all $x \in A$. Hence, by virtue of Theorem 6.1, $\dim(X) \leqslant n$. □

We finish this paper with the following two characterizations of the covering dimension in collectionwise normal spaces.

Theorem 6.4. *For a τ-collectionwise normal space X, closed $B \subset X$ and $n \geqslant 0$, the following conditions are equivalent*:

(a) $\dim_X(X \setminus B) \leqslant n$.

(b) *If $A \subset X$ is a closed subset containing B and Y is a completely metrizable space such that $w(Y) \leqslant \tau$ and $\dim(Y) < \infty$, then every u.s.c. $\theta : A \to \mathcal{C}_{n+1}(Y)$ is a selection for $\varphi|A$ for some u.s.c. $\varphi : X \to \mathcal{C}_{n+1}(Y)$.*

Proof. For (a) $\Rightarrow$ (b) apply Theorem 5.4(b) with $\Phi(x) = Y$, $x \in X$; (b) $\Rightarrow$ (a) follows from Theorem 6.2 because $\dim(Y_{n+2}) = 0 < \infty$. □

Theorem 6.5. *For a countably paracompact and τ-collectionwise normal space X, closed $B \subset X$ and $n \geqslant 0$, the following conditions are equivalent*:

(a) $\dim_X(X \setminus B) \leqslant n$.

(b) *If $A \subset X$ is a closed subset containing B and Y is a completely metrizable space with $w(Y) \leqslant \tau$, then every u.s.c. $\theta : A \to \mathcal{C}_{n+1}(Y)$ is a selection for $\varphi|A$ for some u.s.c. $\varphi : X \to \mathcal{C}_{n+1}(Y)$.*

Proof. Repeat the proof of Theorem 6.4 but now use Theorem 5.4(a) instead of Theorem 5.4(b). □

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